NAG Toolbox for MATLAB

c06lb

1 Purpose

c06lb computes the inverse Laplace transform $f(t)$ of a user-supplied function $\mathbf{f}(s)$, defined for complex s. The function uses a modification of Weeks' method which is suitable when $f(t)$ has continuous derivatives of all orders. The function returns the coefficients of an expansion which approximates $f(t)$ and can be evaluated for given values of t by subsequent calls of c06lc.

2 Syntax

```
[sigma, b, m, acoef, errvec, ifail] = c06lb(f, sigma0, sigma, b, epstol,
'mmax', mmax)
```

3 Description

Given a function $f(t)$ of a real variable t, its Laplace transform $\mathbf{f}(s)$ is a function of a complex variable s, defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \qquad \mathrm{Re}(s) > \sigma_0.$$

Then $f(t)$ is the inverse Laplace transform of $\mathbf{f}(s)$. The value σ_0 is referred to as the abscissa of convergence of the Laplace transform; it is the rightmost real part of the singularities of $\mathbf{f}(s)$.

c06lb, along with its companion c06lc, attempts to solve the following problem:

given a function $\mathbf{f}(s)$, compute values of its inverse Laplace transform $f(t)$ for specified values of t.

The method is a modification of Weeks' method (see Garbow *et al.* 1988a), which approximates $f(t)$ by a truncated Laguerre expansion:

$$\tilde{f}(t) = e^{\sigma t} \sum_{i=0}^{m-1} a_i e^{-bt/2} L_i(bt), \qquad \sigma > \sigma_0, \qquad b > 0$$

where $L_i(x)$ is the Laguerre polynomial of degree i. This function computes the coefficients a_i of the above Laguerre expansion; the expansion can then be evaluated for specified t by calling c06lc. You must supply the value of σ_0, and also suitable values for σ and b: see Section 8 for guidance.

The method is only suitable when $f(t)$ has continuous derivatives of all orders. For such functions the approximation $\tilde{f}(t)$ is usually good and inexpensive. The function will fail with an error exit if the method is not suitable for the supplied function $\mathbf{f}(s)$.

The function is designed to satisfy an accuracy criterion of the form:

$$\left| \frac{f(t) - \tilde{f}(t)}{e^{\sigma t}} \right| < \epsilon_{tol}, \qquad \text{for all } t$$

where ϵ_{tol} is a user-supplied bound. The error measure on the left-hand side is referred to as the **pseudo-relative error**, or **pseudo-error** for short. Note that if $\sigma > 0$ and t is large, the absolute error in $\tilde{f}(t)$ may be very large.

c06lb is derived from the (sub)program MODUL1 in Garbow *et al.* 1988a.

4 References

Garbow B S, Giunta G, Lyness J N and Murli A 1988a Software for an implementation of Weeks' method for the inverse laplace transform problem *ACM Trans. Math. Software* **14** 163–170

Garbow B S, Giunta G, Lyness J N and Murli A 1988b Algorithm 662: A Fortran software package for the numerical inversion of the Laplace transform based on Weeks' method *ACM Trans. Math. Software* **14** 171–176

5 Parameters

5.1 Compulsory Input Parameters

1: **f – string containing name of m-file**

f must return the value of the Laplace transform function $\mathbf{f}(s)$ for a given complex value of s.

Its specification is:

```
[result] = f(s)
```

Input Parameters

1: **s – complex scalar**

The value of s for which $\mathbf{f}(s)$ must be evaluated. The real part of **s** is greater than σ_0.

Output Parameters

1: **result – complex scalar**

The result of the function.

2: **sigma0 – double scalar**

The abscissa of convergence of the Laplace transform, σ_0.

3: **sigma – double scalar**

The parameter σ of the Laguerre expansion. If on entry **sigma** $\leq \sigma_0$, **sigma** is reset to $\sigma_0 + 0.7$.

4: **b – double scalar**

The parameter b of the Laguerre expansion. If on entry **b** $< 2(\sigma - \sigma_0)$, **b** is reset to $2.5(\sigma - \sigma_0)$.

5: **epstol – double scalar**

The required relative pseudo-accuracy, that is, an upper bound on $\left|f(t) - \tilde{f}(t)\right|e^{-\sigma t}$.

5.2 Optional Input Parameters

1: **mmax – int32 scalar**

Default: The dimension of the array **acoef**.

an upper bound on the number of Laguerre expansion coefficients to be computed. The number of coefficients actually computed is always a power of 2, so **mmax** should be a power of 2; if **mmax** is not a power of 2 then the maximum number of coefficients calculated will be the largest power of 2 less than **mmax**.

Suggested value: **mmax** $= 1024$ is sufficient for all but a few exceptional cases.

Default: 1024

Constraint: **mmax** ≥ 8.

5.3 Input Parameters Omitted from the MATLAB Interface

None.

5.4 Output Parameters

1: **sigma – double scalar**

The value actually used for σ, as just described.

2: **b – double scalar**

The value actually used for b, as just described.

3: **m – int32 scalar**

The number of Laguerre expansion coefficients actually computed. The number of calls to user-supplied complex function **f** is $\mathbf{m}/2 + 2$.

4: **acoef(mmax) – double array**

The first **m** elements contain the computed Laguerre expansion coefficients, a_i.

5: **errvec(8) – double array**

An 8-component vector of diagnostic information:

errvec(1)	=	overall estimate of the pseudo-error $\lvert f(t) - \tilde{f}(t) \rvert e^{-\sigma t}$;
	=	**errvec**(2) + **errvec**(3) + **errvec**(4);
errvec(2)	=	estimate of the discretization pseudo-error;
errvec(3)	=	estimate of the truncation pseudo-error;
errvec(4)	=	estimate of the condition pseudo-error on the basis of minimal noise levels in function values;
errvec(5)	=	K, coefficient of a heuristic decay function for the expansion coefficients;
errvec(6)	=	R, base of the decay function for the expansion coefficients;
errvec(7)	=	logarithm of the largest expansion coefficient; and
errvec(8)	=	logarithm of the smallest nonzero expansion coefficient.

The values K and R returned in **errvec**(5) and **errvec**(6) define a decay function KR^{-i} constructed by the function for the purposes of error estimation. It satisfies

$$|a_i| < KR^{-i}, \qquad \text{for } i = 1, 2, \ldots, m.$$

6: **ifail – int32 scalar**

0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Note: c06lb may return useful information for one or more of the following detected errors or warnings.

ifail = 1

On entry, **mmax** < 8.

ifail = 2

The estimated pseudo-error bounds are slightly larger than **epstol**. Note, however, that the actual errors in the final results may be smaller than **epstol** as bounds independent of the value of t are pessimistic.

ifail = 3

Computation was terminated early because the estimate of rounding error was greater than **epstol**. Increasing **epstol** may help.

ifail $= 4$

The decay rate of the coefficients is too small. Increasing **mmax** may help.

ifail $= 5$

The decay rate of the coefficients is too small. In addition the rounding error is such that the required accuracy cannot be obtained. Increasing **mmax** or **epstol** may help.

ifail $= 6$

The behaviour of the coefficients does not enable reasonable prediction of error bounds. Check the value of **sigma0**. In this case, **errvec**(i) is set to -1.0, for $i = 1$ to 5.

When **ifail** ≥ 3, changing **sigma** or **b** may help. If not, the method should be abandoned.

7 Accuracy

The error estimate returned in **errvec**(1) has been found in practice to be a highly reliable bound on the pseudo-error $|f(t) - \tilde{f}(t)|e^{-\sigma t}$.

8 Further Comments

8.1 The Role of σ_0

Nearly all techniques for inversion of the Laplace transform require you to supply the value of σ_0, the convergence abscissa, or else an upper bound on σ_0. For this function, one of the reasons for having to supply σ_0 is that the parameter σ must be greater than σ_0; otherwise the series for $\tilde{f}(t)$ will not converge.

If you do not know the value of σ_0, you must be prepared for significant preliminary effort, either in experimenting with the method and obtaining chaotic results, or in attempting to locate the rightmost singularity of $\mathbf{f}(s)$.

The value of σ_0 is also relevant in defining a natural accuracy criterion. For large t, $f(t)$ is of uniform numerical order $ke^{\sigma_0 t}$, so a **natural** measure of relative accuracy of the approximation $\tilde{f}(t)$ is:

$$\epsilon_{\mathrm{nat}}(t) = \left(\tilde{f}(t) - f(t)\right)/e^{\sigma_0 t}.$$

The function uses the supplied value of σ_0 only in determining the values of σ and b (see below); thereafter it bases its computation entirely on σ and b.

8.2 Choice of σ

Even when the value of σ_0 is known, choosing a value for σ is not easy. Briefly, the series for $\tilde{f}(t)$ converges slowly when $\sigma - \sigma_0$ is small, and faster when $\sigma - \sigma_0$ is larger. However the natural accuracy measure satisfies

$$|\epsilon_{\mathrm{nat}}(t)| < \epsilon_{tol} e^{(\sigma - \sigma_0)t}$$

and this degrades exponentially with t, the exponential constant being $\sigma - \sigma_0$.

Hence, if you require meaningful results over a large range of values of t, you should choose $\sigma - \sigma_0$ small, in which case the series for $\tilde{f}(t)$ converges slowly; while for a smaller range of values of t, you can allow $\sigma - \sigma_0$ to be larger and obtain faster convergence.

The default value for σ used by the function is $\sigma_0 + 0.7$. There is no theoretical justification for this.

8.3 Choice of b

The simplest advice for choosing b is to set $b/2 \geq \sigma - \sigma_0$. The default value used by the function is $2.5(\sigma - \sigma_0)$. A more refined choice is to set

$$b/2 \geq \min_j |\sigma - s_j|$$

where s_j are the singularities of $\mathbf{f}(s)$.

9 Example

```
c06lb_f.m

function [f] = c06lb_f(s)
  f=3.0/(s^2-9.0);
  if isreal(f)
    f=complex(f);
  end
```

```
sigma0 = 3;
sigma = 0;
b = 0;
epstol = 1e-05;
[sigmaOut, bOut, m, acoef, errvec, ifail] = ...
    c06lb('c06lb_f', sigma0, sigma, b, epstol, 'mmax', int32(512))
```

```
sigmaOut =
    3.7000
bOut =
    1.7500
m =
          64
acoef =
    0.4400
   -0.1506
   -0.0614
   -0.0533
   -0.0403
   -0.0311
   -0.0239
   -0.0184
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